

GROUP THEORY 2024 - 25, EXERCISE SHEET 1

Exercise 1. The Permutation Group S_n

- (1) Recall from *structure algébrique* that every element of S_n can be represented as a product of disjoint cycles. We start off with a quick warm-up of the cycle notation.

- (a) Write the following permutation in cycle notation for the group S_5 :

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{pmatrix}$$

- (b) Compute the product $(1\ 3\ 4\ 5)(2\ 6) \cdot (1\ 6\ 5\ 2)(3\ 4)$ in S_6 .
(c) Compute the inverse of the permutation $(1\ 3\ 4\ 5)(2\ 6)$ in the group S_6 .
- (2) Let p be a prime number and let $a \in S_n$ such that $a^p = 1$, then show that a can be written as a product of disjoint p -cycles (that is cycles of length p).
- (3) Recall that a transposition in S_n is a permutation that can be written as $(i\ j)$ with $1 \leq i < j \leq n$.
(a) Show that every permutation in S_n can be written as a product of transpositions.
(b) Let τ_i and π_i be some transpositions in S_n such that

$$\tau_1 \cdot \tau_2 \cdot \dots \cdot \tau_k = \pi_1 \cdot \pi_2 \cdot \dots \cdot \pi_l.$$

Show that l and k have the same parity, that is $l - k = 0 \pmod{2}$.

- (c) Permutations that can be written as a product of an odd (respectively even) number of transpositions are called odd (respectively even) permutations. Show that an m -cycle in S_n is an odd permutation if and only if m is an even integer.

Exercise 2. The Dihedral Group D_{2n}

Recall the definition of the dihedral group.

- (1) Show that the set D_{2n} endowed with the function composition law forms a group. Is it a subgroup of S_n ?
(2) Show that there exists an isomorphism $D_6 \cong S_3$ and deduce that D_6 is not abelian;
(3) Find two elements $r, s \in D_6$ such that every element $x \in D_6$ can be written as a composition of r and s . You showed that D_6 is generated by two elements;
(4) Generalize and find two elements $r, s \in D_{2n}$ such that every element $x \in D_{2n}$ can be written as a composition of r and s . You showed that D_{2n} is generated by two elements. Later on in the course we will see that D_{2n} has a *presentation* with generators r and s ;
(5) Show that D_{2n} is not abelian for all $n \geq 3$;

Exercise 3. *Some more examples of groups.*

- (1) Let $G = \{1, -1, i, -i\}$ with complex multiplication. Show that it forms a group isomorphic to $\mathbb{Z}/4\mathbb{Z}$.
- (2) Let K be the set of all complex numbers z such that $|z| = 1$. Verify that K is a group under complex multiplication.
- (3) Let H be the set of all complex n^{th} roots of unity, that is all solutions to the equation $X^n - 1 = 0$ with $X \in \mathbb{C}$. Prove that G is a group under complex multiplication which is isomorphic to the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

Exercise 4. *Group Actions*

Let $\cdot : G \times X \rightarrow X$ be a group action. For every $x \in X$ we define

$$\text{Stab}_G(x) = \{g \in G \mid g \cdot x = x\} \subseteq G$$

the stabilizer subset of G at $x \in X$.

- (1) Show that $\text{Stab}_G(x)$ is a group and construct an injective group homomorphism

$$i : \text{Stab}_G(x) \rightarrow G.$$

For the following actions, determine its stabilizers groups at the indicated $x \in X$:

- (2) Let $G = S_n$ acting on $X = \{1, \dots, n\}$ with the permutation action. Solve for all $x \in X$.
- (3) Let $G = GL_n(K)$ the group of invertible matrices endowed with matrix multiplication, and $X = M_{n \times n}(K)$ with the multiplication action $A \cdot B = AB$ for $A \in G$ and $B \in X$. Solve for $x = E_{i,j}$ and all $x \in GL_n(K)$;
- (4) With the same group and set as in the previous point, consider the conjugation action $A \cdot B = ABA^{-1}$. Solve for x a scalar matrix, and for x a diagonal matrix with *distinct* entries.
- (5) The left multiplication action $G \times G \rightarrow G$, defined by $g \cdot g' = gg'$; and the conjugation action $g \bullet g' = gg'g^{-1}$. Compute the stabilizers group of $G = S_n$ at $\sigma = (ij)$ a permutation and at $\tau = (ijk)$ a 3-cycle; and of $\mathbb{Z}/n\mathbb{Z}$ at all $a \in \mathbb{Z}/n\mathbb{Z}$.
- (6) Let $G = C_2 = \{1, -1\}$ a multiplicative cyclic group of order 2 and $X = \mathbb{R}$ with the action given by multiplication. Solve for all $x \in X$.

Exercise 5. *Some Geometric Actions*

- (1) Let G be the group of rotations of a cube, show that $G \cong S_4$ by considering the natural action of G on the 4 long diagonals of the cube.
- (2) Recall that the group D_{2n} has a defining action on the set of vertices of a regular n -gon. The aim of this exercise is to show that D_{12} can be realised as a subgroup of S_5 by considering an appropriate geometric action:
Let $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ denote the set of vertices of a regular hexagon in cyclic order. By T_1 and T_2 we denote the triangles with vertex sets $\{a_1, a_3, a_5\}$ and $\{a_2, a_4, a_6\}$ respectively. Also by D_1, D_2, D_3 we denote the long diagonals of the hexagon, that is the line segments $\{a_1, a_4\}, \{a_2, a_5\}, \{a_3, a_6\}$ respectively. We define

$$S := \{T_1, T_2, D_1, D_2, D_3\}.$$

Observe that the natural action of D_{12} on a regular hexagon defines an action on S and show that the corresponding group homomorphism

$$D_{12} \rightarrow S_5$$

is injective.